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The Spectral Action Principle in Noncommutative Geometry and the Superstring

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Abstract

A supersymmetric theory in two-dimensions has enough data to define a non-commutative space thus making it possible to use all tools of noncommutative geometry. In particular, we apply this to the $N = 1$ supersymmetric non-linear sigma model and derive an expression for the generalized loop space Dirac operator, in presence of a general background, using canonical quantization. The spectral action principle is then used to determine a spectral action valid for the fluctuations of the string modes.

It is now generally accepted that at very high energies, the structure of space-time could not adequately be described by a manifold. Quantum fluctuations makes it difficult to define localised points. The most familiar example is string theory where points are replaced by strings, and space-time becomes a loop space [1, 2, 3]. What has been lacking, up to now, are the mathematical tools necessary to realize such spaces geometrically. Fortunately, the recent advances in noncommutative geometry as formulated by Connes [4] makes it possible to tackle such problems. The main advantage in adopting Connes' formulation of noncommutative geometry is that the geometrical data is determined by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an algebra of operators, $\mathcal{H}$ a Hilbert space amd D a Dirac operator acting on $\mathcal{H}$. These ideas have been successfully applied to simple generalizations of space-time such as a product of discrete by continuos spaces . The results are very encouraging in the sense that with a very simple imput one gets all the details of the standard model including the Higgs mechanism, and the unification of the Higgs fields with the gauge fields [5], as well as unification with gravity [6, 7] .

Supersymmetric field theories in two-dimensions have enough data to define noncommutative geometries [8, 9, 10]. In two-dimensions one can have (p, q) supersymmetry as the left and right moving sectors could be split, giving rise to various possibilities. The simplest possibilities are $N = 1$ and $N = \frac{1}{2}$ (i.e. $(1, 1)$ and $(1, 0)$ respectively). A good starting point would be to consider various superconformal field theories and use the noncommutative geometric tools to define geometric objects of interest. This would be fruitful in cases such as orbifold compactifications where many useful data is available to help define the noncommutative geometric space completely. In this letter we shall adopt a slightly different framework where the starting point is the supersymmetric non-linear sigma model in two-dimensions [11], with a general curved target space background. The conserved supersymmetric charges satisfy the supersymmetry algebra [1]. Canonical quantization would then change these charges to Dirac operators over the loop space $\Omega(M)$ where M is a Riemannian spin-manifold. The square of a Dirac operator when restricted to reparametrization invariant configurations gives the Hamiltonian of the system, as an elliptic pseudo-differential operator. These operators could be used to write down a spectral action as a function of the background fields, which gives the low-energy effective action of string theory when the loops are shrunk to points. At high-energies where oscillators are present the full spectral action must be considered.

The plan of this letter is as follows. First we give the essential definitions needed to define a noncommutative space. Next we consider an $N = 1$ supersymmetric non-linear sigma model on a curved background with torsion and construct the corresponding Dirac operator. The algebra is given by the algebra of continous functions on the loop space, and the Hilbert space is the Hilbert space $\mathcal{H}$ of states usually comprising two sectors, the Ramond sector (R) with periodic boundary conditions for fermions and Neveu-Schwarz (NS) with anti-periodic boundary conditions for fermions. These data are then used to define a

spectral action based on the loop space Dirac operators, which will also give the Hamiltonian and momentum operators in two-dimensions. We note that such considerations have been performed before to determine the index of the loop space Dirac operator (elliptic genus) but without torsion [12, 13]. Similar considerations were also performed for the non-linear sigma models for point particles (with or without torsion) [14, 15, 16],where the Hamiltonian of the system was determined. The proposed action satisfies the constraints that it gives at low-energies the string effective action, and the partition function in the limit when the background geometry is flat.

A starting point in defining noncommutative geometry [4] is the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is a $*$ algebra of bounded operators acting on a seperable Hilbert space $(\mathcal{H},\, D)$ is a Dirac operator on $\mathcal{H}$ such that $[D, a]$ is a bounded operator for arbitrary $a \in \mathcal{A}$. A K-cycle $\mathcal{H}, D$ for $\mathcal{A}$ is said to be even if there exists a unitary involution Γ on $\mathcal{H}$ such that $\Gamma a = a\Gamma$ for all $a \in \mathcal{A}$ and $\Gamma D = -D\Gamma$. Given a unital algebra $\mathcal{A}$ one can define the universal differential algebra $\Omega^{\cdot}(\mathcal{A}) = \oplus_{n=0}^{\infty}\Omega^n(\mathcal{A})$ as follows: One sets $\Omega^0(\mathcal{A}) = \mathcal{A}$ and define $\Omega^n(\mathcal{A})$ to be the linear space given by

$$\Omega^n(\mathcal{A}) = \left\{ \sum_i a_0^i da_1^i \cdots da_n^i \;\; : a_j^i \in \mathcal{A}, \forall i, j \right\} \tag{1}$$

where d satisfies Liebnitz rule. We define a representation π of $\Omega^{\cdot}(\mathcal{A})$ on $\mathcal{H}$ by setting

$$\pi\left(\sum_i a_0^i da_1^i \cdots da_n^i \right) = \sum_i \pi(a_0^i)[D, \pi(a_1^i)] \cdots [D, \pi(a_n^i)] \tag{2}$$

Modulo the subtelty of moding the kernel of the representation π out, it is possible to define geometric objects such as distance, metric, connection, curvature and so on [17].

In physics, symmetry plays an important role, and it is important to identify the symmetries associated with a noncommutative space. In gauge theories the relevant symmetries are the internal gauge symmetries, and in general relativity these are diffeomorphisms of the manifold. In noncommtative geometry, it was shown that these symmetries arise naturally as the automorphisms of the algebra $\mathcal{A}$, i.e. $Aut(\mathcal{A})$ [18]. Many Dirac operators associated with special geometries are known. Fluctuations induced under automorphisms of the algebra would change the special Dirac operators to the generic type. These are determined by computing the one-form, $\pi(\rho) = \sum_i a^i[D, b^i]$ so that the new operators are of the form $D + \pi(\rho)$. The new Dirac operator would include information about all the geometric invariants. It was conjectured in [6] that the spectral action describes the dynamics of the fluctuations, and even without knowing the exact form of the functional dependence of the action on D, a lot of information could be extracted about the dynamics. Of course a well defined theory will eventually have its spectral action in terms of a completely determined function.

In a different direction, in Witten's work on the relation between supersymmetry and Morse theory [1], it was shown that the supersymmetry charge of a supersymmetric non-linear sigma model can be viewed as a Dirac operator on an infinite dimensional loop space

$\Omega(M)$ where M is the target space spin-manifold. In the case of $N=1$ supersymmetry there exists two supersymmetry charges Q_+ and Q_- satisfying

$$\begin{aligned}
Q_+^2 &= H + P, \\
Q_-^2 &= H - P, \\
\{Q_+, Q_-\} &= 0
\end{aligned} \tag{3}$$

If in the Hilbert space $\mathcal{H}$ a vacuum state is annihilated by $Q_\pm$ then it is also a vacuum state of the system. Restricting to states satisfying $P=0$ would make the following identifications possible:

$$\begin{aligned}
Q_+ &= d + d^*, \\
Q_- &= i(d - d^*), \\
H &= dd^* + d^*d
\end{aligned} \tag{4}$$

with $d^2 = d^{*2} = 0$. The operators d and d^* are the differential operator and its Hodge dual on M.

By considering different non-linear sigma models such as the $N=\frac{1}{2}$ model corresponding to heterotic strings, one obtains loop geometries which are generalizations of $\Omega(M)$ allowing for gauge fields.

We start with the $N=1$ non-linear sigma model with background fields $G_{\mu\nu}[\Phi]$ and $B_{\mu\nu}[\Phi]$ which are symmetric and antisymmetric in $\mu\nu$ respectively. Here $\Phi(\xi,\theta_+,\theta_-)$ is a superfield with ξ the coordinates on the two-dimensional world sheet. The two-dimensional action is [11]

$$I = \frac{T}{2}\int d^2\xi d\theta_+ d\theta_- \left(G_{\mu\nu}[\Phi] + B_{\mu\nu}[\Phi]\right)\left(D_-\Phi^\mu D_+\Phi^\nu\right) \tag{5}$$

whrere T is the string tension and the component form of the superfield Φ^μ is

$$\Phi^\mu = X^\mu + i\theta_+\psi^{\mu+} - i\theta_-\psi^{\mu-} + i\theta_+\theta_- F^\mu \tag{6}$$

The operators $D_\pm$ are the supersymmetric derivatives (not to be confused with Dirac operators which in this work will be denoted by $Q_\pm$) and are given by

$$\begin{aligned}
D_+ &= \frac{\partial}{\partial\theta_+} - i\theta_+\partial_+ \\
D_- &= \frac{\partial}{\partial\theta_-} - i\theta_-\partial_- \\
\partial_\pm &= \partial_0 \pm \partial_1
\end{aligned} \tag{7}$$

We assume that the the two-dimensional reparametrization invariance has been gauge fixed in the superconformal gauge, and the corresponding superghost system has been added (we will come back to this point later). The coordinates in two dimension are $\xi^0=\tau$ and $\xi^1=\sigma$ with $\tau\in R$ and $\sigma\in[0,2\pi]$. We will comment later on the possibility of having more general backgrounds. The result of expanding the action (5) in component form, after eliminating

the auxiliary fields F^μ and performing the integration over the Grassmann variables is well known, and is given by [11]

$$
\begin{aligned}
I \;=\; & \frac{T}{2}\int d^2\xi\Big[\left(G_{\mu\nu}[X]+B_{\mu\nu}[X]\right)\partial_- X^\mu\partial_+ X^\nu \\
& +i\psi^{a+}\left(\eta_{ab}\partial_- + \omega^+_{\mu ab}\partial_- X^\mu\right)\psi^{b+} + i\psi^{a-}\left(\eta_{ab}\partial_+ + \omega^-_{\mu ab}\partial_+ X^\mu\right)\psi^{b-} \\
& +\frac{1}{2}\psi^{a+}\psi^{b+}\psi^{c-}\psi^{d-}R^+_{cdab}[X] \\
& +\frac{i}{2}\partial_-\left(B_{\mu\nu}[X]\psi^{\mu+}\psi^{\nu+}\right) - \frac{i}{2}\partial_+\left(B_{\mu\nu}[X]\psi^{\mu-}\psi^{\nu-}\right)\Big]
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
R^{\mu\pm}_{\ \nu\rho\sigma} &= \partial_\rho\Gamma^{\mu\pm}_{\sigma\nu} + \Gamma^{\mu\pm}_{\rho\kappa}\Gamma^{\kappa\pm}_{\sigma\nu} - (\rho\leftrightarrow\sigma) && (9)\\
\Gamma^{\mu\pm}_{\nu\rho} &= \Gamma^{\mu}_{\nu\rho} \pm \frac{1}{2}H^{\mu}{}_{\nu\rho} && (10)\\
H_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} && (11)
\end{aligned}
$$

The fermions with tangent indices $a, b, \ldots$ are related to those with curved ones by:

$$\psi^a(\xi) = \psi^\mu[X]e^a_\mu[X] \tag{12}$$

The varriation of the action (8) gives the conserved currents $j_\pm$ [16]

$$j_\pm = T\left(\psi^{\mu\pm}G_{\mu\nu}\partial_\pm X^\nu \pm \frac{i}{6}\psi^{\mu\pm}\psi^{\nu\pm}\psi^{\rho\pm}H_{\mu\nu\rho}\right) \tag{13}$$

Canonical quantization of the bosons give

$$[X^\mu(\sigma,\tau), P_\nu(\sigma',\tau)] = i\delta^\nu_\mu\delta(\sigma-\sigma') \tag{14}$$

while that for the fermions give (after replacing Poisson's brackets with Dirac brackets)

$$\left\{\psi^{a\pm}(\sigma,\tau), \psi^{b\pm}(\sigma',\tau)\right\} = \eta^{ab}\delta(\sigma-\sigma') \tag{15}$$

The momentum $P_\mu(\sigma,\tau)$ is related to the other fields through the equation:

$$P_\mu = T\left(G_{\mu\nu}\partial_0 X^\nu + \frac{i}{2}\psi^{a+}\psi^{b+}\omega^+_{\mu ab} + \frac{i}{2}\psi^{a-}\psi^{b-}\omega^-_{\mu ab}\right) \tag{16}$$

where $\omega^\pm_{\mu ab}$ is related to the spin-connection $\omega_{\mu ab}$ by

$$\omega^\pm_{\mu ab} = \omega_{\mu ab} \pm \frac{1}{2}H_{\mu ab} \tag{17}$$

and the spin-connection is defined through the condition

$$\frac{\delta}{\delta X^\mu}e^a_\nu[X] - \Gamma^\rho_{\mu\nu}[X]e^a_\rho[X] - \omega_\mu{}^a{}_b[X]e^b_\nu[X] = 0 \tag{18}$$

The Dirac-Ramond operators $Q_\pm$ are defined as the integrals of the currents $j_\pm$:

$$Q_\pm = \int d\sigma j_\pm \tag{19}$$

It proves more useful to rotate the fermions to the chiral basis [14]:

$$\begin{aligned} \psi^{a+} &= -\tfrac{1}{\sqrt{2}}\left(\chi^a + \overline{\chi}^a\right) \\ \psi^{a-} &= -\tfrac{i}{\sqrt{2}}\left(\chi^a - \overline{\chi}^a\right) \end{aligned} \tag{20}$$

which will now satisfy the anticommutation relations:

$$\left\{\chi^a(\sigma,\tau), \overline{\chi}^b(\sigma',\tau)\right\} = \frac{1}{2}\eta^{ab}\delta(\sigma-\sigma') \tag{21}$$

It is also useful to define the operators Q and $\overline{Q}$ by

$$\begin{aligned} Q_+ &= Q + \overline{Q} \\ Q_- &= i\left(Q - \overline{Q}\right) \end{aligned} \tag{22}$$

The final result is

$$\begin{aligned} Q &= \int d\sigma \Big(-\frac{i}{\sqrt{T}}\chi^a(\sigma)e^\mu_a[X]\left(\nabla_\mu + \frac{1}{3}H_{abc}\chi^b(\sigma)\chi^c(\sigma)\right) \\ &\qquad + \sqrt{T}\left(\overline{\chi}^a(\sigma)e_{\nu a}[X] - \chi^a(\sigma)e^\mu_a[X]B_{\mu\nu}[X]\right)\frac{dX^\nu}{d\sigma}\Big) \end{aligned} \tag{23}$$

$$\begin{aligned} \overline{Q} &= \int d\sigma \Big(-\frac{i}{\sqrt{T}}\overline{\chi}^a(\sigma)e^\mu_a[X]\left(\nabla_\mu + \frac{1}{3}H_{abc}\overline{\chi}^b(\sigma)\overline{\chi}^c(\sigma)\right) \\ &\qquad + \sqrt{T}\left(\chi^a(\sigma)e_{\nu a}[X] - \overline{\chi}^a(\sigma)e^\mu_a[X]B_{\mu\nu}[X]\right)\frac{dX^\nu}{d\sigma}\Big) \end{aligned} \tag{24}$$

The covariant derivative ∇_μ is defined by

$$\nabla_\mu = \frac{\delta}{\delta X^\mu} + \omega_{\mu ab}[X]\left(\chi^a(\sigma)\overline{\chi}^b(\sigma) + \overline{\chi}^a(\sigma)\chi^b(\sigma)\right) \tag{25}$$

In determining the expressions for Q and $\overline{Q}$ we have used the quantization condition (14) to write

$$P_\mu(\sigma) = -i\frac{\delta}{\delta X^\mu(\sigma)} \tag{26}$$

After a very lengthy calculation, the details of which will be given somewhere else, one can show that

$$Q^2 = \overline{Q}^2 = \frac{1}{2}P \tag{27}$$

where the two-dimensional momentum P is given by

$$P = -i\int d\sigma \frac{dX^\mu}{d\sigma}\nabla_\mu + 2i\int d\sigma \chi_a(\sigma)\frac{D\overline{\chi}^a}{D\sigma} \tag{28}$$

and the covariant derivative $\frac{D}{D\sigma}$ is defined by

$$\frac{D\overline{\chi}^a}{D\sigma} = \frac{d\overline{\chi}^a}{d\sigma} + \frac{dX^\mu}{d\sigma}\omega_\mu{}^a{}_b[X]\overline{\chi}^b(\sigma) \tag{29}$$

A lengthier calculation gives the anticommutator $\{Q,\overline{Q}\} = H$ where

$$\begin{aligned} H \;=\; -\frac{1}{2T}\int d\sigma \Big[& \Big(\nabla^a\nabla_a + \omega_b{}^{ab}[X(\sigma)]\nabla_a + 4\chi^a(\sigma)\overline{\chi}^b(\sigma)\chi^c(\sigma)\overline{\chi}^d(\sigma)R_{abcd}[X(\sigma)]\Big) \\ & +\frac{2}{3}\left(\chi^a(\sigma)\overline{\chi}^b(\sigma)\overline{\chi}^c(\sigma)\overline{\chi}^d(\sigma) + \overline{\chi}^a(\sigma)\chi^b(\sigma)\chi^c(\sigma)\chi^d(\sigma)\right)\nabla_a H_{bcd}[X(\sigma)] \\ & +\left(\chi^b(\sigma)\chi^c(\sigma) + \overline{\chi}^b(\sigma)\overline{\chi}^c(\sigma)\right)H^a{}_{bc}[X(\sigma)]\nabla_a \\ & +\frac{1}{3}\Big(\chi^a(\sigma)\chi^b(\sigma)\overline{\chi}^c(\sigma)\overline{\chi}^d(\sigma) + \chi^a(\sigma)\overline{\chi}^b(\sigma)\overline{\chi}^c(\sigma)\chi^d(\sigma) \\ & \qquad + \overline{\chi}^a(\sigma)\overline{\chi}^b(\sigma)\chi^c(\sigma)\chi^d(\sigma)\Big) H_{ab}{}^e[X(\sigma)]H_{ecd}[X(\sigma)] \\ & -2iT\left(\chi_a\frac{D\chi^a}{D\sigma} + \overline{\chi}_a\frac{D\overline{\chi}^a}{D\sigma}\right) \\ & -2iT\left(\chi^a\frac{D\chi^b}{D\sigma} + \overline{\chi}^a\frac{D\overline{\chi}^b}{D\sigma}\right)B_{ab}[X(\sigma)] \\ & -2iTe_a^\mu[X(\sigma)]e_b^\nu[X(\sigma)]\left(\chi^b(\sigma)\overline{\chi}^a(\sigma) + \overline{\chi}^b(\sigma)\chi^a(\sigma)\right)\left(\nabla_\rho B_{\mu\nu} - \nabla_\nu B_{\mu\rho}\right)\frac{dX^\rho}{d\sigma} \\ & +2iTB_{\mu\nu}\frac{dX^\nu}{d\sigma}\nabla^\mu - 2i\left(\chi^a(\sigma)\chi^b(\sigma) + \overline{\chi}^a(\sigma)\overline{\chi}^b(\sigma)\right)H_{ab}{}^c[X(\sigma)]B_{c\nu}[X(\sigma)]\frac{dX^\nu}{d\sigma} \\ & -T^2\left(G_{\mu\nu}[X(\sigma)] + B_{\mu\kappa}[X(\sigma)]B^\kappa{}_\nu[X(\sigma)]\right)\frac{dX^\mu}{d\sigma}\frac{dX^\nu}{d\sigma}\Big] \end{aligned} \tag{30}$$

Throughout eq (30), curved indices are changed to tangent ones with $e_a^\mu[X(\sigma)]$ and its inverse. It is clear that the Hamiltonian is a second order elliptic operator in loop space, and that P is the generator of reparametrizations on the circle. The supersymmetry generators $Q_\pm$ are related to Q and $\overline{Q}$ by (22) . The target space coordinates $X^\mu(\sigma,\tau)$ are taken at time $\tau = 0$. The dependence on time is governed by the equation:

$$X^\mu(\sigma,\tau) = e^{-\tau H}X^\mu(\sigma,0)\; e^{\tau H} \tag{31}$$

The advantage of adopting the canonical quantization is that one can use the Fourier expansion of $X^\mu(\sigma)$. In the case of the closed superstring we have (at $\tau = 0$):

$$X^\mu(\sigma) = X_0^\mu + \sum_{n>0}\frac{1}{\sqrt{\pi nT}}\left(a_n^\mu\cos n\sigma + \tilde{a}_n^\mu\sin n\sigma\right) \tag{32}$$

and the momentum P_μ of (26) is expressed as

$$P_\mu = -i\left(\frac{1}{2\pi}\frac{\delta}{\delta X_0^\mu} + \sum_{n>0}\sqrt{\frac{nT}{\pi}}\left(\frac{\delta}{\delta a_n^\mu}\cos n\sigma + \frac{\delta}{\delta\tilde{a}_n^\mu}\sin n\sigma\right)\right) \tag{33}$$

Similarly, the fermions $\chi^a(\sigma)$ and $\overline{\chi}^a(\sigma)$ can be expanded in terms of oscillators:

$$\begin{aligned} \chi^a(\sigma) &= \tfrac{1}{\sqrt{2\pi}}\sum_{r\in Z_0+\phi}\left(c_r\cos r\sigma + d_r\sin r\sigma\right) \\ \overline{\chi}^a(\sigma) &= \tfrac{1}{\sqrt{2\pi}}\sum_{r\in Z_0+\phi}\left(\overline{c}_r\cos r\sigma + \overline{d}_r\sin r\sigma\right) \end{aligned} \tag{34}$$

where $\phi = 0$ for Ramond (R) boundary conditions (i.e. periodic) and $\phi = \frac{1}{2}$ for Neveu-Schwarz (NS) boundary conditions. The quantization conditions on the fermions imply that the only non-vanishing anti-commutators are

$$\begin{aligned} \{c_r^a, \overline{c}_s^b\} &= 2\delta_{rs}\eta^{ab} \quad r, s \neq 0 \\ \{d_r^a, \overline{d}_s^b\} &= 2\delta_{rs}\eta^{ab} \end{aligned} \tag{35}$$

The fermionic zero modes occur only in the R-sector, which satisfy the anticommutation relations:

$$\{c_0^a, \overline{c}_0^b\} = \eta^{ab} \tag{36}$$

Therefore both $c_0^a + \overline{c}_0^a$ and $i(c_0^a - \overline{c}_0^a)$ generate Clifford algebras, and give rise to creation and annihilation operators for the vacuum state.

This is not the full story. In obtaining the action, the superparametrization invariance has been fixed, and the superghost part must be added to compensate for fixing the metric and gravitino. This is well known and given by (see e.g. [19]):

$$I^{(\text{ghost})} = -\frac{1}{2\pi}\int d^2\xi d\theta_+ d\theta_- \left(BD_-C + \overline{B}D_+\overline{C}\right) \tag{37}$$

where the fields B and C and have the component expansions:

$$\begin{aligned} B &= \beta + i\theta_+ b \qquad & \overline{B} &= \overline{\beta} - i\theta_-\overline{b} \\ C &= c + i\theta_+\gamma \qquad & \overline{C} &= \overline{c} - i\theta_-\overline{\gamma} \end{aligned} \tag{38}$$

The supercurrents are:

$$\begin{aligned} j_+^{(\text{ghost})} &= -c\partial_+\beta + \tfrac{1}{2}\gamma b - \tfrac{3}{2}\partial_+ c\beta \\ j_-^{(\text{ghost})} &= -\overline{c}\partial_-\overline{\beta} + \tfrac{1}{2}\overline{\gamma}\overline{b} - \tfrac{3}{2}\partial_-\overline{c}\overline{\beta} \end{aligned} \tag{39}$$

The fields b, c, β, γ satisfy the quantization conditions:

$$\begin{aligned} \{\, b(\sigma,\tau), c(\sigma',\tau)\} &= 2\pi\delta(\sigma - \sigma') \\ [\beta(\sigma,\tau), \gamma(\sigma',\tau)] &= 2\pi\delta(\sigma - \sigma') \end{aligned} \tag{40}$$

One can define the ghost Dirac-Ramond operators by

$$Q_\pm^{(\text{ghost})} = \frac{1}{2}\int d\sigma j_\pm^{(\text{ghost})} \tag{41}$$

which will satisfy

$$(Q_\pm^{(\text{ghost})})^2 = H^{(\text{ghost})} \pm P^{(\text{ghost})} \tag{42}$$

The Hamiltonian and momenta of the ghost system do not interact with the rest, but simply add up, allowing for this part to be computed seperately, as it is independent of the background fields.

From the above analysis it should be clear that the operators Q and $\overline{Q}$ togother with the Hilbert space and algebra of functions over the loop space, define the geometry. The only other operators that we need are the K-cycle involutions Γ and $\overline{\Gamma}$ which satisfy

$$\{\Gamma, Q\} = 0 = [\overline{\Gamma}, Q] \tag{43}$$

and similarly for $\overline{Q}$:

$$\{\overline{\Gamma}, \overline{Q}\} = 0 = [\Gamma, \overline{Q}] \tag{44}$$

In the case of the superstring, these are the fermion numbers defined for left and right movers (or here associated with Q and $\overline{Q}$). From reparametrization invariance one must insure that physical states satisfy $P = 0$. The spectral action must be a function of of the operators Q and $\overline{Q}$ as well as Γ and $\overline{\Gamma}$. Therefore, and on general grounds, a good candidate for the spectral action that will describe the dynamics of the background fields is given by [6]

$$\mathrm{Tr}\ f\left(Q, \overline{Q}, \Gamma, \overline{\Gamma}\right) \tag{45}$$

where the trace is taken over all states in the Hilbert space. Finding the correct form of the function f is a difficult task. However, even without knowing the form of the function, the dependence on the background fields is given in terms of geometric invariants. To every order in a heat-kernel type expansion (through a Fourier or Melin transform) invariants would enter multiplied by an overall numerical factor (Fourier components of the function). This was the case when the spectral principle was applied to the noncommutative space defining the standard model. There, the whole action was determined, up to terms not higher than second order in curvature, in terms of the first three geometric invariants in the heat kernel expansion. The coefficients of these terms were chosen to fit known gauge and Higgs couplings implying some relations among them.

Fortunately, in the situation considered here, the theory is known in two limits. First when the background metric is flat, and the other is for the zero modes where one gets the low-energy field theory limit. First we consider the case when the background geometry is flat. For flat backgrounds, the Hamiltonian (30) simplifies to

$$H = -\frac{1}{2T}\int d\sigma[\frac{\delta}{\delta X^{\mu}}\frac{\delta}{\delta X_{\mu}} - T^{2}\frac{dX^{\mu}}{d\sigma}\frac{dX_{\mu}}{d\sigma} - 2iT(\chi_{a}\frac{d\chi^{a}}{d\sigma} + \overline{\chi}^{a}\frac{d\overline{\chi}_{a}}{d\sigma})] \tag{46}$$

This Hamiltonian could be expressed in terms of creation and annihilation operators. The path integral expression of the one-loop amplitude,is related to the partition function [14], in the case when the two-dimensional surface is a torus. The result is modular invariant, and therefore consistent (free of anomalies) if the dimension of the target space is ten. We also have to set $T = \frac{1}{4\pi l_s^2}$ where l_s is the string length scale. Also to project the non-physical states out (or equivalently, require modular invariance when the two-dimensional surface has genus greater than or equal to two) one must have the partition function [20]

$$I = \int \frac{d\tau d\overline{\tau}}{\tau_2^2}\mathrm{Tr}\left|\sum_{NS\oplus R}\left(e^{2\pi i(\tau Q^2}(-1)^{\epsilon}(1-\Gamma)\right)\right|^2 \tag{47}$$

where $\epsilon = 0$ for the NS sector, and $\epsilon = 1$ for the Ramond sector over the states in the trace. This action, has space-time supersymmetry as can be verified by counting the number of fermionic and bosonic states (massive as well as massless) and showing they are the same.

The parameter $\tau = \tau_1 + i\tau_2$ is the modular parameter (although τ was used up to now as the two-dimensional time). The total partition function, including the ghosts is

$$\int \frac{d\tau d\overline{\tau}}{\tau_2^2} \int \frac{d^8p}{(2\pi)^8} e^{-2\pi p^2 \tau_2} \left| \frac{1}{2\eta(\tau)^4} \left(\theta_3^4(0|\tau) - \theta_4^4(0|\tau)^4 - \theta_2^4(0|\tau)^4 - \theta_1^4(0|\tau) \right) \right|^2 \tag{48}$$

and as expected, because of supersymmetry, the partition function vanishes. The ghost contributions cancel the contributions of two bosonic and two fermionic coordinates. Since the superghost part is independent of the background, these contributions would be the same even in a curved background. The difficulty is to compute the spectral action in an arbitrary background including the dilaton, and the space-time supersymmetric vacuum so that a space-time gravitino background, as well as a two and three forms would be included. This is not an easy problem to solve since this will make space-time supersymmetry explicit without invoking the Green-Schwarz superstring and κ symmetry [19]. For here we shall limit our considerations to the background we started with (which is not the most general, and can be perturbed to more general backgrounds by transformations which are automorphisms of the algebra $\Omega(M)$). To compute the spectral action in an arbitrary background is a very complicated. We shall only determine the lowest order terms in a perturbative expansion. One starts by splitting the dependence of the fields in the partition function in terms of zero modes and oscillators. In the NS-sector there are no fermionic zero modes and the coordinates $X^\mu(\sigma)$ have a constant part X_0^μ . The Hamiltonian of the zero modes is

$$H_{\mathrm{NS}}^0 = -[\nabla_0^a \nabla_{0a} + \omega_{0b}{}^{ab} \nabla_{0a}] \tag{49}$$

In the R-sector, there are fermionic zero modes χ_0^a and the zero modes Hamiltonian is

$$\begin{aligned} H_{\mathrm{R}}^0 \;&=\; [-\nabla_0^a \nabla_{0a} + \omega_{0b}{}^{ab} \nabla_{0a} + 4\chi_0^a \overline{\chi}_0^b \chi_0^c \overline{\chi}_0^d R_{abcd}^0 \\ &+\frac{2}{3}(\chi_0^a \overline{\chi}_0^b \overline{\chi}_0^c \overline{\chi}_0^d + \overline{\chi}_0^a \chi_0^b \chi_0^c \chi_0^d + \overline{\chi}_0^a \chi_0^b \chi_0^c \overline{\chi}_0^d) \nabla_{0a} H_{0bcd} \\ &+2(\chi_0^b \chi_0^c + \overline{\chi}_0^b \overline{\chi}_0^c) H_{0bca} \nabla_{0a} \\ &+\frac{1}{3}(\chi_0^a \chi_0^b \overline{\chi}_0^c \overline{\chi}_0^d + \chi_0^a \overline{\chi}_0^b \overline{\chi}_0^c \chi_0^d + \chi_0^a \overline{\chi}_0^b \overline{\chi}_0^c \overline{\chi}_0^d + \overline{\chi}_0^a \overline{\chi}_0^b \chi_0^c \chi_0^d) H_{0ab}{}^{e} H_{0ecd}] \end{aligned}$$

With these operators it is possible to use the heat kernel expansion to evaluate the trace of the exponential in the form [21]

$$\operatorname{Tr}(e^{-\tau_2 \mathcal{P}}) = \sum_{n=0}^{\infty} a_n(\mathcal{P})\, \tau_2^{\frac{n-D}{2}}$$

where $a_n(\mathcal{P})$ are the Seeley-de Wit coefficients corresponding to the operator $\mathcal{P}$ and $D = 10$ is the dimension of the target manifold. Using the results of heat kernel expansion for a general second order operator, one finds the following results [21]:

$$\operatorname{Tr}(e^{-\tau_2 H_{\mathrm{NS}}^0}) = \frac{a_0(H_{\mathrm{NS}}^0)}{\tau_2^5} + \frac{a_2(H_{\mathrm{NS}}^0)}{\tau_2^4} + \cdots$$

where

$$a_0(H^0_{\mathrm{NS}}) = \frac{1}{(2\pi)^5}\int d^{10}X_0\sqrt{G[X_0]} \tag{50}$$

and the center of mass coordinates X_0^μ become coordinates on the manifold. The next term in the expansion is

$$a_2(H^0_{\mathrm{NS}}) = \frac{1}{(2\pi)^5}\int d^{10}X_0\sqrt{G[X_0]}(\frac{1}{6}R[X_0]) \tag{51}$$

Similarly, for the R-sector, we have $a_0(H^0_{\mathrm{R}}) = a_0(H^0_{\mathrm{NS}})$ while for the next term a_2 we have

$$a_2(H^0_{\mathrm{R}}) = \frac{1}{(2\pi)^5}\int d^{10}X_0\sqrt{G}(-\frac{1}{12}R[X_0] - \frac{1}{24}H_{0\mu\nu\rho}H_0^{\mu\nu\rho}) \tag{52}$$

Higher orders in the expansion would involve higher curvature terms, and will receive contributions from the oscillator parts. This can be done in a perturbative expansion using normal coordinates. To lowest orders, and for the a_0 terms, this is given by an expansion of the terms appearing in (48) . This implies that the coefficient of the a_0 term vanishes which is the cosmological constant. For the a_2 terms we have to expand , to lowest order in τ, $(\theta_3^4 - \theta_4^4)$ multiplying the NS-sector and $-(\theta_2^4 + \theta_1^4)$ multiplying the R-sector. The net contributions to lowest order is proportional to

$$\int d^{10}X_0\sqrt{G[X_0]}(\frac{1}{4}R[X_0] + \frac{1}{24}H_{0\mu\nu\rho}H_0^{\mu\nu\rho}) \tag{53}$$

Comparing this with the superstring effective action at low energies [22, 23] we find that they are identical to this order. This is extremely encouraging, as we had no free parameters to adjust. The challenging problem that remains is to find a closed expression for the spectral action as a function of the background geometry in analogy with the calculation of the elliptic genus in [12] where modular invariance plays an important role. Of course the effective superstring action has more terms depending on the dilaton, three-form, vector and gravitinos. It is possible to include the dilaton by adding to the non-linear sigma model the Weyl breaking term

$$\int d^2\xi d\theta_+ d\theta_- \ \mathrm{sdet}E \ \phi[\Phi] \ \mathcal{R}^\pm \tag{54}$$

where ϕ is the background dilaton, and $\mathcal{R}\pm$ is the super-curvature in two-dimensions and sdet$E$ is the super-determinant of the super-zweibein. As mentioned earlier, it is more difficult to include in a curved background in a covariant way, the spinors on the target manifold. The only known way is the Green-Schwarz formulation which is studied in a light-cone gauge and is not explicitely world-sheet supesymmetric (for an effective action derivation see e.g. [24]). In a noncommutative formulation it is quite important to have explicit world sheet supersymmetry as this is necessary to derive the supercharges whose integrals give the Dirac-Ramond operators. These points and other details are now under study.

In conclusion, we have shown that superstring non-linear sigma models provide natural examples of noncommutative geometry spaces as developed by Connes. The tools of

noncommutative geometry are available to study these spaces geometrically. Recent ideas proposed in noncommutative geometry for writing a spectral action describing the dynamics of geometric fields (metric, gauge fields, forms, ...) are used and shown to give correct answers in some known limits. It remains to find, in analogy with the case of the elliptic genus, closed form for this action in terms of geometric invariant. Details of the results presented here will appear somewhere else.

References

[1] E. Witten, J. Diff. Geom. **17** (1982) 661.

[2] E. Witten, In: "Anomalies, geometry and topology" Eds W. Bardeen and and A. White, World Scientific 1985, p61.

[3] M. J. Bowick and S. G. Rajeev, Phys. Rev. Lett. **58** (1986) 535

[4] A. Connes, "Noncommutative geometry", Academic Press, 1994; Pub. Math. IHES **62** (1983) 44.

[5] M. Dubois-Viollette, R. Kerner and J. Madore, J. Math. Phys. **31** (1990) 316;
A. Connes and J. Lott, Nucl. Phys. (Proc. Supp.) B18 (1990) 29; In: Proceedings of the 1991 Cargese summer conference, edited by J. Fröhlich et al (Plenum, New York 1992);
D. Kastler, Rev. Math. Phys. **5** (1993) 477;
A. H. Chamseddine, G. Felder and J. Fröhlich, Phys.Lett. **B296** (1992) 109; Nucl. Phys. **B395** (1993) 672.

[6] A. H. Chamseddine and A. Connes, Phys. Rev. Lett. **77** (1996) 4868; hep-th/9606001 to appear in Commun. Math. Phys.

[7] B. Iochum, D. Kastler and T. Schucker, hep-th/9607158

[8] A. H. Chamseddine and J. Fröhlich, In: Yang Festschrift, Eds C. S. Lui and S. -F. Yau, International Press, Boston, 1995.

[9] J. Fröhlich and K. Gawedzki, CPM Proceedings lecture notes, **7** (1994) 57.

[10] J. Fröhlich, O. Grandjean and A. Recknagel, ETH-TH/96-45.

[11] L. Alvarez-Gaumé and D. Z. Freedman, Commun. Math. Phys. **80** (1981)443.

[12] E. Witten, Commun. Math. Phys. **109** (1987) 525.

[13] Commun. Math. Phys. **112** (1987) 75.

[14] L. Alvarez-Gaumé, Commun. Math. Phys. **90** (1983) 161.

[15] A. C. Davies, A. J. Macfarlane, P. C. Popat and J. W. van Holten, J. Phys. A: Math. Gen. **17** (1984) 2945.

[16] H. W. Braden, Ann.Phys. **171** (1986) 433.

[17] A. H. Chamseddine, G. Felder and J. Fröhlich, Commun. Math. Phys. **155** (1993) 205.

[18] A. Connes, Commun. Math. Phys. **182** (1996) 155.

[19] M. B. Green J. H. Schwarz and E. Witten, "Superstring Theory" Cambridge University Press, 1987.

[20] N. Seiberg and E. Witten, Nucl. Phys. **B276** (1986) 272.

[21] P. B. Gilkey, "Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem" Publish or Perish Inc, 1984.

[22] A. H. Chamseddine, Nucl. Phys. **B185** (1981) 403.

[23] I. Campbell and P. West, Nucl. Phys. **B243** (1984) 112.

[24] W. Lerche, Nucl. Phys. **B308** (1988) 102.